

ON THE DOUBLED TETRUS

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ABSTRACT. The “tetras” is a member of a family of hyperbolic 3-manifolds with totally geodesic boundary, described by Paoluzzi-Zimmerman, which also contains W.P. Thurston’s “tripus.” Each member of this family has a branched cover to B^3 , branched over a certain tangle T . This map on the tripus has degree three, and on the tetras degree four. We describe a cover of the double of the tetras, itself a double across a closed surface, which fibers over the circle.

This paper describes some features of a fibered cover of a certain 3-manifold, related to a family of compact hyperbolic 3-manifolds $M_{n,k}$ with totally geodesic boundary defined by Paoluzzi-Zimmermann [14]. A well known member of this family is $M_{3,1}$, Thurston’s “tripus” [17, Ch. 3]. Here we consider the “tetras” $M_{4,1}$ (thanks to Richard Kent for naming suggestions).

If M is an oriented manifold with boundary, let \bar{M} be a copy of M with orientation reversed, and define the *double* of M to be $DM = M \cup_{\partial} \bar{M}$, where the gluing isometry $\partial M \rightarrow \partial \bar{M}$ is induced by the identity map.

Theorem 0.1. *There is a cover $p: D\tilde{M} \rightarrow DM_{4,1}$ of degree 6, where $D\tilde{M}$ is a double across the closed surface $p^{-1}(\partial M_{4,1})$ of genus 13, and $D\tilde{M}$ fibers over S^1 with fiber \tilde{F} , a closed surface of genus 19.*

The manifold $D\tilde{M}$ above is the first hyperbolic 3-manifold which we know to be both fibered and a double across a closed surface. Non-hyperbolic fibered doubles are easily constructed, for instance by doubling the exterior of a fibered knot across its boundary torus, but producing a fibered hyperbolic double is more subtle problem. In such a manifold the doubling surface — necessarily with genus at least 2 — does not itself admit a fibering and must thus have points of tangency with any fibering, which in particular cannot be invariant under the doubling involution.

The strategy of proof for Theorem 0.1 is motivated by the fact that the doubled tetras branched covers S^3 , branched over the link L of Figure 1, which is the Montesinos link $L(1/3, 1/2, -1/2, -1/3)$. Thurston observed that given a branched cover $M_n \rightarrow M$, branched over a link L , virtual fiberings of M transverse to the preimage of L may be pulled back to virtual fiberings of M_n . This is also used in [4], which is where we encountered it, and [2]. We record a version of this observation as Proposition 1.2, and apply it here with Proposition 1.3 to prove Theorem 0.1.

When M' is homeomorphic to the Cartesian product of a closed surface F with S^1 , and L is a link contained in a disjoint union of fibers, Proposition 1.3 produces new fiberings of M' transverse to L under certain circumstances. We state and prove Proposition 1.3 in Section 1, along with Proposition 1.2. In Section 2 we describe the manifolds $M_{n,k}$ constructed in [14], and their branched covers to the 3-ball. The branched cover $DM_{4,1} \rightarrow S^3$, which results from doubling, factors

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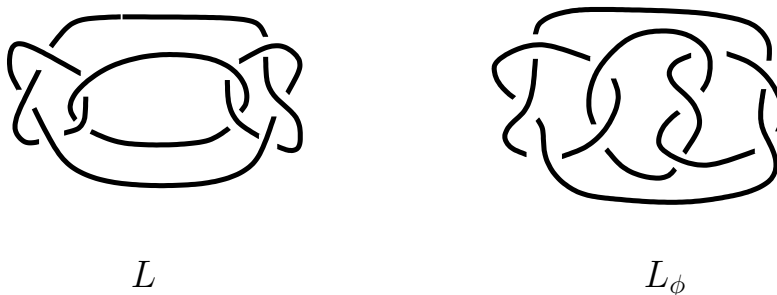


FIGURE 1. $DM_{n,k}$ and $D_\phi M_{n,k}$ n -fold cover S^3 , branched over L and L_ϕ .

through a double branched cover $DM_2 \rightarrow S^3$, branched over L . Such a manifold is well known to have the structure of a Seifert fibered space ([13], cf. [7]). In Section 3, we describe the Seifert fibered structure on DM_2 and construct a cover $p': DM' \rightarrow DM_2$ which satisfies the hypotheses of Proposition 1.3. An application of Proposition 1.2 completes the proof of Theorem 0.1.

Steven Boyer and Xingru Zhang previously obtained results about virtual fiberings of Montesinos links and their branched covers (see [2]) using a similar strategy. These imply that the doubled tetrus is virtually fibered; in particular, [2, Theorem 1.7] applies more generally than our independently discovered Proposition 1.3. An advantage of Proposition 1.3, when it applies, is that it produces an explicit description of a fiber surface, which we use in Theorem 0.1 to obtain the extra information about the genus of \tilde{F} .

It is well known that $M_{n,k}$ has a minimal-genus Heegaard splitting of genus n , obtained by attaching a single one-handle to $\partial M_{n,k}$. (Ushijima classified such splittings in [18, Theorem 2.8].) Thus the tetrus $M_{4,1}$ has a Heegaard splitting with genus 4, yielding an amalgamated Heegaard splitting of $DM_{4,1}$ with genus 5. The preimage in \tilde{DM} gives the following corollary of Theorem 0.1.

Corollary 0.2. *\tilde{DM} has a weakly reducible Heegaard splitting of genus 25 associated to $p^{-1}(\partial M_{4,1})$, and one of genus 39 associated to \tilde{F} .*

It would be interesting to know the minimal genus of a fiber surface for \tilde{DM} , for the above discussion shows that if this is greater than twelve, the minimal genus Heegaard splitting of \tilde{DM} is not associated to a fibering. Such examples are nongeneric, according to work of Souto [15, Theorem 6.2] and Biringer [6]. See also [5] for a survey of results about degeneration.

The *twisted double* $D_\phi M_{n,k}$ in the proposition is obtained by gluing $M_{n,k}$ to its mirror image via an isometry ϕ of the boundary, the lift to $M_{n,k}$ of the mutation producing the link L_ϕ of Figure 1. Our original motivation for proving Theorem 0.1 was Proposition 0.3 below, which describes arithmeticity of the doubles and twisted doubles for $n = 3$ and 4.

Proposition 0.3. *The doubles $DM_{3,k}$ ($k = 0$ or 1) and $DM_{4,k}$ ($k = 1$ or 3) are nonarithmetic. On the other hand, $D_\phi M_{3,k}$ and $D_\phi M_{4,k}$ are arithmetic.*

This may be verified using Snappea and Snap, together with the descriptions of $DM_{n,k}$ and $D_\phi(M_{n,k})$ as n -fold covers of L and L_ϕ above. For each n , $DM_{n,k}$

and $D_\phi M_{n,k}$ contain a totally geodesic surface identical to the totally geodesic boundary of $M_{n,k}$. In the cases $n = 3$ and 4 , it follows from arithmeticity of the twisted doubles that this surface is arithmetic. (This can also be discerned directly from a polyhedral decomposition.) Using Proposition 4.1 of [11], one obtains the following.

Corollary 0.4. *The doubled tetrus has a nested, cofinal family of regular covers with respect to which it has property τ .*

Work of Abert-Nikolov ([1]) concerning rank and Heegaard genus has drawn attention to virtually fibered manifolds which satisfy the conclusion of Corollary 0.4. The doubled tetrus is the first closed nonarithmetic hyperbolic 3-manifold that we knew to have both of these properties, although work of Agol [3] shows that some nonarithmetic right-angled reflection groups, for instance in the Löbell polyhedron $L(7)$, have them both as well.

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1. VIRTUALLY FIBERING BRANCHED COVERS

We will be concerned in this paper with branched coverings. The *standard* k -fold cyclic branched covering of the disk D^2 to itself is the quotient map which identifies each point $z \in D^2 \subset \mathbb{C}$ with points of the form $ze^{2\pi i \frac{j}{k}}$, $0 \leq j < k$. For a 3-manifold M , an n -fold branched covering $q: M_n \rightarrow M$, branched over a link $L \subset M$, is characterized by the property that L has a closed regular neighborhood $\mathcal{N}(L)$, with *exterior* $\mathcal{E}(L) = \overline{M - \mathcal{N}(L)}$, such that

- (1) On $\mathcal{E}_n \doteq q^{-1}(\mathcal{E}(L))$, q restricts to a genuine n -fold covering map.
- (2) Each component V of $q^{-1}(\mathcal{N}(L))$ has a homeomorphism to $D^2 \times S^1$ so that $q|_V$ is the product of the standard k -fold branched cover with the identity map $S^1 \rightarrow S^1$, for some $k > 1$ dividing n .

Remark. It might be more accurate to allow the map in condition 2 to be the product of the k -fold branched covering of D^2 with a nontrivial covering of S^1 to itself. Since we will not encounter examples with this property here, we restrict our attention to the simpler setting.

Now suppose $p': M' \rightarrow M$ is a genuine g -fold covering space, and let $\mathcal{E}' = (p')^{-1}(\mathcal{E}(L)) \subset M'$ be the associated cover of $\mathcal{E}(L)$. The group $\pi(L) \doteq \pi_1(\mathcal{E}(L))$ has subgroups Γ_n , and Γ' , corresponding to the covers of $\mathcal{E}(L)$ by \mathcal{E}_n and \mathcal{E}' , respectively. Below we record an elementary observation about Γ' .

Fact. Let $V \cong D^2 \times S^1$ be a component of $\mathcal{N}(L)$ and take $\mu = \partial D^2 \times \{y\} \subset \mathcal{E}(L)$ for some $y \in S^1$. If $[\mu]$ represents the homotopy class of μ in $\pi(L)$, then $[\mu] \in \Gamma'$.

This holds because $p'|_{\mathcal{E}'}$ extends to a covering map on M' ; hence each component of the preimage of μ bounds a lift to M' of the inclusion map $D^2 \times \{y\} \hookrightarrow M$. We will call *meridians* curves of the form $\partial D^2 \times \{y\} \subset \partial \mathcal{E}(L)$. Take $\tilde{\Gamma} = \Gamma_n \cap \Gamma'$, and let $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ be the associated covering space, factoring through the restriction of q to \mathcal{E}_n and of p' to \mathcal{E}' . We will say that $\tilde{\mathcal{E}}$ *completes the diamond*.

Lemma 1.1. *Let $p': M' \rightarrow M$ be a g -fold cover and $q_n: M_n \rightarrow M$ an n -fold branched cover, branched over a link $L \subset M$. Let $\mathcal{E}(L)$ be the exterior of L and \mathcal{E}' , \mathcal{E}_n its covers associated to M' and M_n , respectively. If $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}(L)$ completes the diamond, then \tilde{p} extends to a map $\tilde{p}: \tilde{M} \rightarrow M$ such that $\tilde{p} = p' \circ q = q_n \circ p$ for a branched cover $q: \tilde{M} \rightarrow M'$ and a cover $p: \tilde{M} \rightarrow M_n$.*

Proof. Since $\tilde{\mathcal{E}}$ corresponds to the subgroup $\tilde{\Gamma} = \Gamma_n \cap \Gamma' < \pi(L)$, there are coverings $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}_n$ and $q: \tilde{\mathcal{E}} \rightarrow \mathcal{E}'$ such that $\tilde{p} = q_n \circ p = p' \circ q$.

Let $V \cong D^2 \times S^1$ be a component of $\mathcal{N}(L)$, let $\lambda = \{x\} \times S^1$ for some $x \in D^2$, and let $\mu = \partial D^2 \times \{y\} \in \mathcal{E}(L)$ for some $y \in S^1$. Taking $[\mu] \in \pi(L)$ to represent the homotopy class of μ , we have $[\mu] \in \Gamma'$, so $[\mu]^k \in \tilde{\Gamma}$ if and only if $[\mu]^k \in \Gamma_n$. On the other hand, condition 2 in the characterization of branched coverings implies that each component of the preimage of λ in \mathcal{E}_n maps homeomorphically to λ under q_n . Thus $[\lambda] \in \Gamma_n$, so $[\lambda]^j \in \tilde{\Gamma}$ if and only if $[\lambda]^j \in \Gamma'$.

Let T be a component of $\partial \tilde{\mathcal{E}}$ which maps under \tilde{p} to ∂V , and let $\tilde{\mu}$ and $\tilde{\lambda}$ be components on T of the preimages of μ and λ , respectively. Then by the above, the map p restricts to a homeomorphism of $\tilde{\mu}$ onto its image, and $\deg p|_{\tilde{\lambda}} = \deg p|_T$. On the other hand, q has the same degree on T as it does on $\tilde{\mu}$, and it restricts on λ to a homeomorphism.

Let $\tilde{\mathcal{E}}(\tilde{\mu})$ be the manifold obtained from $\tilde{\mathcal{E}}$ by Dehn filling along $\tilde{\mu}$. That is $\tilde{\mathcal{E}}(\tilde{\mu})$ is the quotient of $\tilde{\mathcal{E}} \sqcup \tilde{V}$, where $\tilde{V} \cong D^2 \times S^1$, by a homeomorphism $\partial D^2 \times S^1 \rightarrow T$ taking $\partial D^2 \times \{y\}$ to $\tilde{\mu}$ for some $y \in S^1$. We may assume without loss of generality that this homeomorphism takes some $\{x\} \times S^1$ to $\tilde{\lambda}$. Then p extends to a covering map $\tilde{\mathcal{E}}(\tilde{\mu}) \rightarrow \mathcal{E}_n(p(\tilde{\mu}))$, sending \tilde{V} to its image by the product of the identity map with the $(\deg p|_T)$ -fold cover of S^1 to itself. Likewise, q extends to a branched cover $\tilde{\mathcal{E}}(\tilde{\mu}) \rightarrow \mathcal{E}'(q(\tilde{\mu}))$.

We now take \tilde{M} to be the manifold obtained from $\tilde{\mathcal{E}}$ by Dehn filling each boundary component along the preimage of the corresponding meridian. It follows from the paragraph above that p extends to the manifold obtained from \mathcal{E}_n by filling boundary components along preimages of meridians. By definition, this is M_n . Similarly, q extends to a branched covering from \tilde{M} to M' , establishing the lemma. \square

Thurston used a version of Proposition 1.2 below to show the reflection orbifold in a right-angled dodecahedron is virtually fibered (cf. [16]); this fact is also used by Boyer-Zhang [2, Cor 1.4]. Our version explicitly describes a fibered cover.

Proposition 1.2. *Suppose $p': M' \rightarrow M$ is a g -fold cover and $q_n: M_n \rightarrow M$ an n -fold branched cover, branched over a link $L \subset M$. If M' fibers over S^1 with fibers transverse to $(p')^{-1}(L)$, then the manifold \tilde{M} supplied by Lemma 1.1 fibers over S^1 in such a way that $q: \tilde{M} \rightarrow M'$ is fiber-preserving.*

Proof. Since $p^{-1}(L)$ is transverse to the fibering of M , for each component V of $\mathcal{N}(L)$, a homeomorphism to $D^2 \times S^1$ may be chosen so that after an ambient isotopy of the fibering, for any component \tilde{V} of $(p')^{-1}(V)$, each fiber intersects \tilde{V} in a collection of disjoint disks of the form $D^2 \times \{y\}$ for $y \in S^1$. Then \mathcal{E}' inherits a fibering from M' with the property that each fiber intersects the boundary in a collection of meridians.

By definition, each curve on $\partial \tilde{\mathcal{E}}$ which bounds a disk in \tilde{M} is a component of the preimage of a meridian of \mathcal{E} . Hence the fibering which $\tilde{\mathcal{E}}$ inherits from \mathcal{E}' by

pulling back using q extends to a fibering of \widetilde{M} , which q maps to that of M' by construction. \square

We will encounter the following situation: M' is the trivial F -bundle over S^1 for some closed surface F , homeomorphic to $F \times I / ((x, 1) \sim (x, 0))$, and $(p')^{-1}(L)$ consists of simple closed curves in disjoint copies of F . Here $I = [0, 1]$. Let $\pi: M' \rightarrow F$ be projection to the first factor. The second main result of this section describes a property of the collection $\pi((p')^{-1}(L))$ which allows a fibering of M' to be found satisfying the hypotheses of Proposition 1.2.

Proposition 1.3. *Let $M = F \times I / (x, 0) \sim (x, 1)$, and suppose $L = \{\lambda_1, \dots, \lambda_m\}$ is a link in M such that for each j there exists $t_j \in I$ with $\lambda_j \subset F \times \{t_j\}$, and $t_j \neq t_{j'}$ for $j \neq j'$. Suppose there is a collection of disjoint simple closed curves $\{\gamma_1, \dots, \gamma_n\}$ on F , each transverse to $\pi(\lambda_j)$ for all j , with the following properties. For each j , there is an i such that $\pi(\lambda_j)$ intersects γ_i , and a choice of orientation of the γ_i and all curves of L may be fixed so that for any i and j , γ_i and $\pi(\lambda_j)$ have equal algebraic and geometric intersection numbers. Then M has a fibering transverse to L .*

The other fiberings needed to prove this theorem may be found by *spinning* annular neighborhoods of the γ_i in the fiber direction. We first saw this technique in [10].

Definition. Let $M = F \times I / (x, 0) \sim (x, 1)$ be the trivial bundle, and let γ be a simple closed curve in F . Let A be a small annular neighborhood of γ , and fix a marking homeomorphism $\phi: S^1 \times I \rightarrow A$. We define the fibration obtained by spinning A in the fiber direction to be

$$F_A(t) = ((F - A) \times \{t\}) \bigcup \Phi_t(S^1 \times I),$$

where $\Phi_t(x, s) = (\phi(x, s), \rho(s) + t)$, for t between 0 and 1. Here we take $\rho: I \rightarrow I$ to be a smooth, nondecreasing function taking 0 to 0 and 1 to 1, which is constant on small neighborhoods of 0 and 1 and has derivative at least 1 on $[1/4, 3/4]$.

Given a collection of disjoint simple closed curves $\gamma_1, \dots, \gamma_n$, one analogously produces a new fibration $F_{A_1, \dots, A_n}(t)$ by spinning an annular neighborhood of each in the fiber direction.

Suppose λ is a simple closed curve in F which has identical geometric and algebraic intersection numbers with the core of each annulus A_i in such a collection; that is, an orientation of λ is chosen so that each oriented intersection with the core of each A_i has positive sign.

Lemma 1.4. *Let λ be such a curve, embedded in M by its inclusion into $F \times \{t_0\}$, $t_0 \in (0, 1)$. There is an ambient isotopy which moves λ to be transverse to the fibration $F_{A_1, \dots, A_n}(t)$, and which may be taken to be supported in an arbitrarily small neighborhood of $F \times \{t_0\}$.*

Proof. λ may be isotoped in F so that its intersection with the A_i is of the form $(\{x_1\} \times I) \sqcup \dots \sqcup (\{x_k\} \times I)$ for some collection $\{x_1, \dots, x_k\}$ of points in their cores. For reference fix a Riemannian metric on F in which the A_i are isometrically embedded with their natural product metric, and choose a smooth unit-speed parametrization $\lambda(t)$ ($t \in I$) so that $\lambda([1/4, 3/4]) = \{x_1\} \times [1/4, 3/4]$. For fixed small $\epsilon > 0$, we embed λ in M with the aid of a map $h_\epsilon: I \rightarrow I$, defined as follows. Let h'_ϵ be a smooth bump function which takes the value $-\epsilon$ on $[0, 1/4]$ and

$[3/4, 1]$, is increasing on $[1/4, 3/8]$ and decreasing on $[5/8, 3/4]$, takes the value 2ϵ on $[3/8, 5/8]$, and has integral equal to 0. Then define h_ϵ by

$$h_\epsilon(s) = t_0 + \int_0^s h'_\epsilon,$$

and let $\lambda_\epsilon(s) = (\lambda(s), h_\epsilon(s))$.

At any point of M , the parametrization of M as $F \times I / (x, 1) \sim (x, 0)$, gives a natural decomposition of the tangent space. We call *horizontal* the tangent planes to F , and let \mathbf{t} denote the vertical vector pointing upward. In the complement of the vertical tori determined by the $A_i \times I$, the new fiber surface $F_{A_1, \dots, A_n}(t)$ has horizontal tangent planes, for each t . Since the intersection of λ_ϵ with this region is contained in $\lambda_\epsilon([0, 1/4] \cup [3/4, 1])$, its tangent vector in this region has \mathbf{t} -component $-\epsilon$. Hence intersections in this region between λ_ϵ and copies of the fiber surface F_{A_1, \dots, A_n} are transverse.

For points in A_i , consider the vertical plane spanned by \mathbf{t} and the tangent vector to the I -factor of A_i . Tangent vectors to λ_ϵ at points which lie in $A_i \times I$ lie in this plane with slope between $-\epsilon$ and 2ϵ , possibly greater than $-\epsilon$ only between $1/4$ and $3/4$. On the other hand, the intersection of the tangent plane to F_{A_1, \dots, A_n} intersects the vertical plane in a line with slope greater than or equal to 0, and greater than or equal to 1 on $[1/4, 3/4]$. Thus as long as $2\epsilon < 1$, any intersection of λ_ϵ with a copy of F_{A_1, \dots, A_n} in these regions is transverse as well. The original embedding of λ may clearly be moved to λ_ϵ by a small ambient isotopy, and since λ_ϵ is transverse to each $F_{A_1, \dots, A_n}(t)$, this proves the lemma. \square

Proof of Proposition 1.3. Let $\{\gamma_1, \dots, \gamma_n\}$ be a collection satisfying the hypotheses, and let $\{A_i\}$ be a collection of disjoint annular regular neighborhoods of the γ_i in F . By the lemma above, each member λ_j of L may be moved by an ambient isotopy to be transverse to the fibration obtained by spinning each A_i in the fiber direction. Since the components of L lie in disjoint fibers of the original fibration, these isotopies may be taken to have disjoint supports. Then the inverse of their composition, applied to the fibration obtained by spinning A in the fiber direction, produces a new fibration which is transverse to L . \square

2. INTRODUCING THE TETRUS

In this section and the next, we will frequently encounter branched coverings of the form $q: M_n \rightarrow M$, where M_n and M are manifolds with nonempty boundary. In this case, the branch locus may have components which are properly embedded arcs in M . If T is the branch locus, we require that the regular neighborhood $\mathcal{N}(T)$ have the property that on a component V of $q^{-1}(\mathcal{N}(T))$ projecting to a neighborhood of an arc component of T , there is a homeomorphism $V \rightarrow D^2 \times I$ such that q is modeled by the product of the k -fold branched cover $D^2 \rightarrow D^2$ with the identity map on I . With the same requirement on circle components of T , and $\mathcal{E}(T)$ defined as before, we remark that Lemma 1.1 holds verbatim in this context.

Thurston constructed a hyperbolic manifold with totally geodesic boundary, which he called the “tripus,” from two hyperbolic truncated tetrahedra in Chapter 3 of his notes [17]. A description of the tripus as the complement of a genus two handlebody embedded in S^3 may be found there. In [14], Paoluzzi-Zimmermann generalized this construction, constructing for each $n \geq 3$ and k between 0 and $n - 1$ with $(2 - k, n) = 1$ a hyperbolic manifold $M_{n,k}$ with geodesic boundary, for

which the tripus is $M_{3,1}$. Ushijima extended Thurston's description of the tripus as a handlebody complement in S^3 to show that each $M_{n,1}$, $n \geq 3$, is homeomorphic to the exterior — that is, the complement of a regular neighborhood — of Suzuki's Brunnian graph θ_n [18]. In particular, the tetrus $M_{4,1}$ is the exterior of the graph θ_4 pictured in Figure 2.

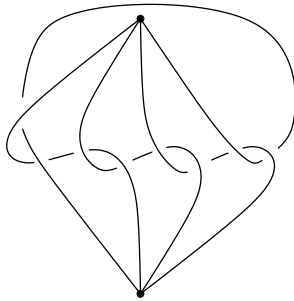


FIGURE 2. θ_4

There is an order-4 automorphism of the tetrus visible in the figure as a rotation through a vertical axis intersecting θ_4 only in its two vertices. The quotient of $M_{4,1}$ by the group that this automorphism generates yields a branched cover $q_{4,1}: M_{4,1} \rightarrow B^3$, branched over the tangle T pictured in Figure 3. In fact Paoluzzi-Zimmermann describe, for each $n \geq 3$ and k with $0 \leq k < n$ and $(2-k, n) = 1$, an n -fold branched cover $q_{n,k}: M_{n,k} \rightarrow B^3$, branched over T (see [14, Figures 4 & 5]). The quotient map $q_{n,k}$ may be realized by a local isometry to an *orbifold* O_n with geodesic boundary, with underlying topological space B^3 and singular locus T with strings of cone angle $2\pi/n$. We summarize Paoluzzi-Zimmermann's description of the orbifold fundamental group of O_n and its relationship with the fundamental groups of the $M_{n,k}$ in the theorem below; this collects various results in [14].

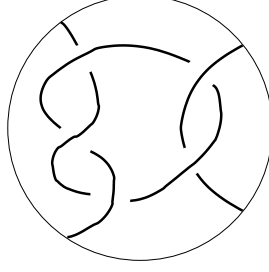
Theorem (Paoluzzi-Zimmermann). *For each $n \geq 3$, the orbifold fundamental group of O_n is presented as*

$$E_n \cong \langle X_n, H_n \mid H_n^n = (H_n X_n H_n X_n^{-2})^n = 1 \rangle.$$

Each elliptic element of E_n is conjugate to exactly one of H_n or $H_n X_n H_n X_n^{-2}$. For $(2-k, n) = 1$, the fundamental group $G_{n,k} := \pi_1(M_{n,k})$ is the kernel of the projection $\pi_k: E_n \rightarrow \mathbb{Z}_n = \langle h_n \rangle$ given by $\pi_k(X_n) = h_n^k$ and $\pi_k(H_n) = h_n$.

Here we have departed from the notation of Paoluzzi-Zimmermann in distinguishing between presentations of the orbifold fundamental group of O_n for different n , so that Paoluzzi-Zimmermann's " x " is replaced above by our " X_n " and similarly for " h " in their presentation for E_n in [14, p. 120]. Also, we have renamed the generator of \mathbb{Z}_n to h_n .

In view of Proposition 0.1 of the previous section, it is important that we have a description of the fundamental group of the exterior of T in B^3 . Let $\mathcal{N}(T)$ be a regular neighborhood of T in B^3 . Then $\mathcal{N}(T)$ has two components, each homeomorphic to $D^2 \times I$ in such a way that its intersection with T is sent to $\{(0,0)\} \times I$ and its intersection with ∂B^3 to $D^2 \times \{0,1\}$. Then take $\mathcal{E}(T) = \overline{B^3 - \mathcal{N}(T)}$, the exterior of T in B^3 , and let $\pi(T) = \pi_1(\mathcal{E}(T))$. We refer by a

FIGURE 3. A tangle T in the ball B^3

meridian of T to a curve on $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ of the form $\partial D^2 \times \{y\}$, for $y \in I$. Below we summarize some facts about $\pi(T)$, which may be found for instance in joint work with Eric Chesebro [8, §2].

Lemma 2.1. *The group $\pi(T)$ is free on generators x and h . In $\pi(T)$, the meridians of T are represented by h and $hxhx^{-2}$, and the four-holed sphere $\partial B^3 \cap \mathcal{E}(T)$ is represented by $\Lambda = \langle h, hxhx^{-2}, (xhx)h^{-1}(xhx)^{-1} \rangle$ in $\pi(T)$.*

Using Lemma 2.1, we reinterpret the Theorem of Paoluzzi-Zimmermann below in a way that matches our treatment of branched covers in Section 1.

Lemma 2.2. *For each $n \geq 3$ and k with $(2-k, n) = 1$, let $\Gamma_{n,k}$ be the kernel of the map $\pi_{n,k}: \pi(T) \twoheadrightarrow \mathbb{Z}_n = \langle h_n \rangle$ given by $x \mapsto h_n^k$, $h \mapsto h_n$, and let $q_{n,k}: \mathcal{E}_{n,k} \rightarrow \mathcal{E}(T)$ be the cover corresponding to $\Gamma_{n,k}$. Then $q_{n,k}$ extends to the branched covering $q_{n,k}: M_{n,k} \rightarrow B^3$ described by Paoluzzi-Zimmermann, where $M_{n,k}$ is obtained from $\mathcal{E}_{n,k}$ by filling each component of the preimage of $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ with a copy of $D^2 \times I$.*

Proof. The homomorphism $\pi_{n,k}$ described in the lemma takes the elements h and $hxhx^{-2}$ to h_n and $(h_n)^{2-k}$, respectively, each of which generates \mathbb{Z}_n when $2-k$ is relatively prime to n . Since the meridians of $\mathcal{E}(T)$ are represented by h and $hxhx^{-2}$, it follows that each meridian has connected preimage in $\mathcal{E}_{n,k}$, represented in $\Gamma_{n,k}$ by h^n and $(hxhx^{-2})^n$, respectively.

If U is a component of $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$, then it is homeomorphic to $\partial D^2 \times I$, and by the paragraph above $(q_{n,k})^{-1}(U)$ is a connected n -fold cover of U , modeled by the product of the n -fold cover $\partial D^2 \rightarrow \partial D^2$ with the identity map $I \rightarrow I$. Thus after filling each component of $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ and its preimage under $q_{n,k}$ with a cylinder $D^2 \times I$, $q_{n,k}$ extends to a branched cover modeled on the cylinders by the product of the standard n -fold branched cover $D^2 \rightarrow D^2$ with the identity map $I \rightarrow I$.

By our descriptions of $\mathcal{N}(T)$ and $\mathcal{E}(T)$, the image of this branched cover is homeomorphic to B^3 , with branching locus T . We claim that the domain is homeomorphic to $M_{n,k}$. There is a map from $\pi(T)$ onto the orbifold group E_n given by sending x and h to X_n and H_n , respectively. Then $\pi_{n,k}$ factors as this projection followed by the map π_k defined by Paoluzzi-Zimmermann. The description of E_n thus implies that $G_{n,k}$ is the quotient of $\Gamma_{n,k}$ by the normal closure of $\{h^n, (hxhx^{-2})^n\}$, and the claim follows. \square

The *double* branched cover $M_2 \rightarrow B^3$, branched over T , was not addressed in [14] since it does not admit the structure of a hyperbolic manifold with totally

geodesic boundary. In fact, it is homeomorphic to the trefoil knot exterior, and we will describe its Seifert fibered structure and the preimage of L in detail in Section 3. Below we give a description consistent with that of Lemma 2.2.

Lemma 2.3. *Let Γ_2 be the kernel of the map $\pi_2: \pi(T) \rightarrow \mathbb{Z}_2 = \{0, 1\}$ given by $x \mapsto 1$, $h \mapsto 1$, and let $q_2: \mathcal{E}_2 \rightarrow \mathcal{E}(T)$ be the cover corresponding to Γ_2 . Then q_2 extends to the unique twofold branched cover $q_2: M_2 \rightarrow B^3$, after filling components of the preimage of $\partial\mathcal{N}(T) \cap \mathcal{E}(T)$ with copies of $D^2 \times I$.*

Proof. Let $q: M \rightarrow B^3$ be a twofold branched cover with branch locus T . The associated cover $q: \mathcal{E} \rightarrow \mathcal{E}(T)$ corresponds to a subgroup $\Gamma < \pi(T)$ which is of index 2 and hence normal. Each element of $\pi(T)$ representing a meridian of T must map nontrivially under the quotient $\pi(T) \rightarrow \pi(T)/\Gamma_2 \simeq \mathbb{Z}_2$, since q branches nontrivially over each component of T . By the description in Lemma 2.1, it follows that h and h^2x^{-1} map nontrivially, hence that each of h and x map to the generator. Thus the only twofold branched cover of B^3 , branched over T , is $q_2: M_2 \rightarrow B^3$ as described in the lemma. \square

The uniqueness of $q_2: M_2 \rightarrow B^3$ implies that each $q_{n,k}: M_{n,k} \rightarrow B^3$, where n is even, factors through q_2 . We will take advantage of this fact below.

3. VIRTUALLY FIBERING THE DOUBLED TETRUS

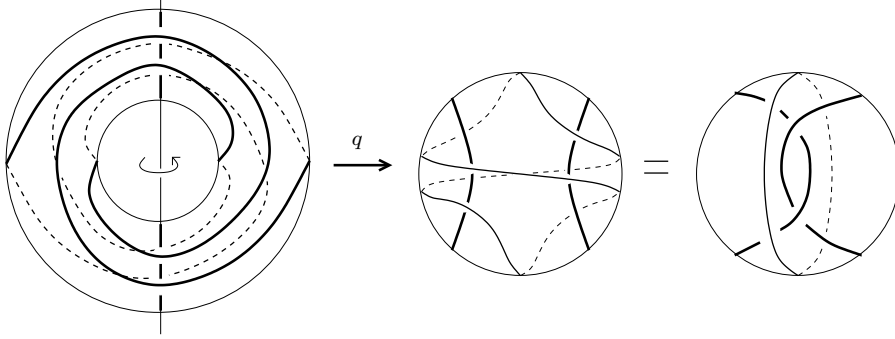
To construct a fibered cover for $DM_{4,1}$ using the methods of Section 1, we first describe a trivially fibered cover $p': M' \rightarrow M_2$ and in it, the preimage of the tangle T of Figure 3. This uses a description of T as a sum of *rational tangles*, introduced by Conway [9]. T is represented as $30 + 20$ in Conway's notation (see [9, Fig. 1,2,3]), where 30 and 20 associate to the rational numbers $1/3$ and $1/2$, respectively. We will refer to T as the *Montesinos tangle* $T(1/3, 1/2)$, referring to Montesinos' construction [13] of twofold branched covers of S^3 , branched over links built as sums of rational tangles. Below we give an ad hoc version of this construction which suits our purposes.

Let $V = D^2 \times S^1$ be the solid torus, embedded in \mathbb{C}^2 as the cartesian product of the unit disk in \mathbb{C} with its boundary, and oriented as a product of the standard orientation on D^2 with the boundary orientation S^1 inherits as ∂D^2 . Define the *complex conjugation-induced* involution of V by $(z, w) \mapsto (\bar{z}, \bar{w})$. The fixed set is $S = \{(r, \pm 1) \mid r \in [-1, 1]\}$, a disjoint union of two arcs properly embedded in V . The quotient map $q: V \rightarrow V/((z, w) \sim (\bar{z}, \bar{w})) \cong B^3$ is a twofold branched covering with branching locus S . Each meridian disk $D^2 \times \{\pm 1\}$ of V is mapped by q to a disk in B^3 which determines an isotopy rel endpoints between an arc of $q(S)$ and an arc on ∂B^3 . That is, $q(S)$ is the trivial two-string tangle B^3 .

A rational number p/q in lowest terms determines a Seifert fibering of V , with an *exceptional fiber* parametrized by $\gamma_0(t) = (0, e^{2\pi i t})$, $t \in I$, and *regular fibers* parametrized by

$$(1) \quad \gamma_z(t) = (ze^{2\pi i \cdot pt}, e^{2\pi i \cdot qt}), \quad t \in I,$$

for $z \in D^2 - \{0\}$. Then z and w in D^2 determine the same fiber if and only if $w = ze^{2\pi i k/q}$ for some $k \in \mathbb{Z}$. Hence for any $w \in S^1$, the quotient map sending each fiber to a point restricts on $D^2 \times \{w\}$ to a q -fold branched covering, branched at the origin. We let $V_{p/q}$ denote V equipped with this fibering, and divide $\partial V_{p/q}$ into annuli $A_{p/q}$ and $B_{p/q}$, parametrized as follows. Define a model annulus $A =$

FIGURE 4. The double branched cover of the rational tangle $1/2$.

$I \times I / (x, 0) \sim (x, 1)$, inheriting a “vertical” fibering by circles from arcs $\{x\} \times I$, a “horizontal” fibering from arcs of the form $I \times \{y\}$, and an orientation from the standard orientation on $I \times I$. Define $\phi_{p/q}: A \rightarrow \partial V_{p/q}$ by

$$\phi_{p/q}(x, y) = \left(e^{2\pi i \left(\frac{1-2x}{4q} \right)} e^{2\pi i \cdot p y}, e^{2\pi i \cdot q y} \right).$$

Choose a and b such that $ap + bq = 1$, and define $\psi_{p/q}: A \rightarrow \partial V_{p/q}$ by

$$\psi_{p/q}(x, y) = \left(e^{2\pi i \left(\frac{2x+1}{4q} \right)} e^{2\pi i \cdot p \left(y - \frac{a}{2q} \right)}, e^{2\pi i \cdot q \left(y - \frac{a}{2q} \right)} \right).$$

Then $\phi_{p/q}$ and $\psi_{p/q}$ have the following properties.

- (1) Taking $A_{p/q} \doteq \phi_{p/q}(A)$ and $B_{p/q} \doteq \psi_{p/q}(A)$, we have $A_{p/q} \cup B_{p/q} = \partial V_{p/q}$, and $A_{p/q} \cap B_{p/q} = \phi_{p/q}(\partial A) = \psi_{p/q}(\partial A)$.
- (2) Each of $\phi_{p/q}$ and $\psi_{p/q}$ takes a vertical fiber of A to a Seifert fiber of $V_{p/q}$ and a horizontal fiber to a closed arc in $\partial D^2 \times \{y\}$ for some $y \in S^1$.
- (3) Giving $A_{p/q}$ and $B_{p/q}$ the boundary orientations from $V_{p/q}$, $\phi_{p/q}$ reverses and $\psi_{p/q}$ preserves orientation.
- (4) The complex conjugation-induced involution on $V_{p/q}$ commutes with the map $(x, y) \mapsto (1-x, 1-y)$ under each of $\phi_{p/q}$ and $\psi_{p/q}$. In particular, each of $A_{p/q}$ and $B_{p/q}$ contains two points of $S \cap \partial V_{p/q}$, and $(1, 1) \in A_{p/q}$.

The map q determines a double branched cover of $V_{1/2}$ to the ball, branched over the rational tangle $1/2$, as illustrated in Figure 4. In the figure, the two parallel simple closed curves comprising $A_{1/2} \cap B_{1/2}$ are drawn on $\partial V_{1/2}$, projecting to the boundary of the indicated disk on B^3 . A similar picture holds for the double branched cover of $V_{1/3}$ to the $1/3$ rational tangle. An appeal to Property 4 of the parametrizations above thus yields the following lemma.

Lemma 3.1. *Define $M_2 = V_{1/3} \cup_{\phi_2 \psi_3^{-1}} V_{1/2}$. There is a branched cover $q_2: M_2 \rightarrow B^3$, branched over T , which restricts on each V_i to q .*

Property 2 of the parametrizations ψ_3 and ϕ_2 implies that M_2 inherits the structure of a Seifert fibered space from $V_{1/3}$ and $V_{1/2}$. The lemma below describes a foliation of M_2 by surfaces, each meeting each Seifert fiber transversely.

Lemma 3.2. *Let $H_m = D^2 \times \{e^{2\pi i \frac{m-1}{2}}\} \subset V_3$, $m = 0, 1$, and $S_n = D^2 \times \{e^{2\pi i \frac{n}{3}}\} \subset V_2$, $n = 0, 1, 2$. Then $F = (\bigcup H_m) \cup (\bigcup S_n)$ is a connected surface homeomorphic*

to a one-holed torus, which is a fiber in the fibration of M_2 that restricts on $V_{1/3}$ or $V_{1/2}$ to the foliation by disks $D^2 \times \{y\}$. A map $\sigma: F \rightarrow F$ is determined by the following combinatorial data: $\sigma(H_m) = H_{1-m}$ for $m = 0, 1$, and $\sigma(D_n) = D_{n+1}$ for $n = 0, 1, 2$ (take $n + 1$ modulo three), so that $M \cong F \times [0, 1]/((x, 0) \sim (\sigma(x), 1))$.

Proof. By Property 2 of $\phi_{1/2}$ and $\psi_{1/3}$, for each $y \in S^1$, the gluing map $\phi_{1/2}\psi_{1/3}^{-1}$ takes components of $\partial D^2 \times \{y\} \cap B_{1/3} \subset V_{1/3}$ to components of $\partial D^2 \times \{y'\} \cap A_{1/2} \subset V_{1/2}$, for y' determined by y . It follows that the foliations of $V_{1/3}$ and $V_{1/2}$ by disks of the form $D^2 \times \{y\}$ join in M_2 to yield a foliation by surfaces. We take F to be the surface in this foliation containing $D^2 \times \{1\}$ in $V_{1/3}$, and illustrate its combinatorics in Figure 5.

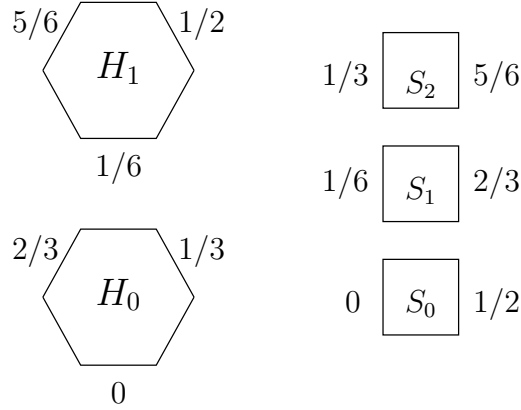


FIGURE 5. The surface F in M_2 .

We depict the disks H_m as hexagons, since each intersects each of $A_{1/3}$ and $B_{1/3}$ in three arcs of its boundary. Applying $\psi_{1/3}^{-1}$ to a component of $H_m \cap B_{1/3}$ yields an arc of the form $I \times \{h\}$ for some $h \in I$; in the figure, we have labeled each arc of $H_m \cap B_{1/3}$ by the corresponding h . Each “square” S_n intersects $A_{1/2}$ in two arcs of its boundary, labeled in Figure 5 by the height of their images under $\phi_{1/2}^{-1}$. We assign each H_m or S_n the standard orientation from D^2 , and picture them in Figure 5 with the orientation inherited from the page. Then each labeled edge of H_m is identified to that of S_n with the same label, in orientation-reversing fashion, by $\phi_{1/2}\psi_{1/3}^{-1}$. Their union F is now easily identified as a one-holed torus.

Since the Seifert fibers of $V_{1/3}$ and $V_{1/2}$ are transverse to the disks $D^2 \times \{y\}$, Seifert fibers of M_2 transversely intersect each surface in the foliation described above. There is a quotient map π , taking M_2 to a closed one-manifold (that is, S^1), determined by crushing each surface in the foliation described above to a point. Then F is the preimage under π of a point in S^1 , so cutting M_2 along F yields a surface bundle over I , necessarily of the form $F \times I$. With F oriented as prescribed above, the normal orientation to F in M_2 is the upward direction along Seifert fibers parameterized as in (1). The “first return map” $\sigma: F \rightarrow F$ is prescribed as follows: from $x \in F$, move in the normal direction along the Seifert fiber through x until it

again intersects F ; the point of intersection is $\sigma(x)$. It describes a monodromy for the description of M_2 as a bundle over S^1 .

From the description of σ and the fiber-preserving parameterizations $\psi_{1/3}$ and $\phi_{1/2}$, we find that for a point x on an arc labeled by h in Figure 5, $\sigma(x)$ is the corresponding point on the arc labeled by $h + 1/6$ (modulo 1). This models the behavior of $\sigma(x)$ on the regular fibers. The intersection of F with the singular fiber in $V_{1/3}$ is the disjoint union of the centers of H_0 and H_1 , which are thus interchanged by σ . An analogous description holds for the intersection of F with the singular fiber in $V_{1/2}$, yielding the description of σ in the statement of the lemma. Now since $V_{1/3}$ is cut by its intersection with F into two cylinders, and $V_{1/2}$ is cut into three, and these are identified along vertical annuli in their boundaries by $\phi_{1/2}\psi_{1/3}^{-1}$ to form M_2 cut along F , the description of M_2 as a fiber bundle with fiber F follows.

By construction, the components of $q_2^{-1}(T)$ in M_2 intersect each of $V_{1/3}$ and $V_{1/2}$ in the fixed set of the complex conjugation-induced involution. In $V_{1/3}$, the component $[-1, 1] \times \{-1\}$ is contained in H_0 , and $[-1, 1] \times \{1\} \subset H_1$. In $V_{1/2}$, the component $[-1, 1] \times \{1\}$ is contained in S_0 , and $\phi_{1/2}\psi_{1/3}^{-1}$ takes its endpoints to endpoints of the components in $V_{1/3}$. \square

Note that $q_2^{-1}(T)$ intersects each of $V_{1/3}$ and $V_{1/2}$ in the fixed locus of the complex conjugation-induced involution. Since the components of this locus lie in the disks $D^2 \times \{\pm 1\}$, it follows that $q_2^{-1}(T)$ lies in fibers of the fibration $M_2 \rightarrow S^1$ described in Lemma 3.2. The description of Lemma 3.2 also implies that σ is periodic with order 6. Therefore M_2 has a sixfold cover $p': M' \rightarrow M_2$ which is trivially fibered; that is, $M' \cong F \times S^1$, such that components of $(q_2 \circ p')^{-1}(T)$ lie in disjoint fibers.

The uniqueness property of Lemma 2.3 implies that the manifolds M_2 described there and in Lemma 3.1 are homeomorphic as branched covers. In particular, the map $q_{4,1}: M_{4,1} \rightarrow B^3$ described in Lemma 2.2 factors through q_2 described in Lemma 3.1. Let $q_{2,1}: M_{4,1} \rightarrow M_2$ be the map such that $q_{4,1} = q_{2,1} \circ q_2$. Then $q_{2,1}$ is a twofold branched cover, branched over $(q_2)^{-1}(T)$.

Recall, from the first paragraph of Section 2, that Lemma 1.1 applies in the context of manifolds with nonempty boundary.

Lemma 3.3. *Let $\tilde{p}: \tilde{M} \rightarrow M_2$ be the map produced by Lemma 1.1, such that $\tilde{p} = p' \circ q = q_{2,1} \circ p$ for a branched cover $q: \tilde{M} \rightarrow M'$ and a cover $p: \tilde{M} \rightarrow M_{4,1}$. Then p has degree 6 and $\partial\tilde{M}$ is connected, with genus 13.*

Proof. Recall that \tilde{p} extends the covering map $\tilde{p}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}_2$, obtained by completing the diamond of covers $q_{2,1}: \mathcal{E}_{4,1} \rightarrow \mathcal{E}_2$ and $p': \mathcal{E}' \rightarrow \mathcal{E}_2$, after filling along components in $\tilde{\mathcal{E}}$ of the preimage of meridians of \mathcal{E}_2 . Lemma 2.3 implies that the map $\pi_2: \pi(T) \rightarrow \mathbb{Z}_2$ takes the elements h and $hxhx^{-2}$ representing the meridians of T to the generator. Hence each meridian of T has connected preimage, the corresponding meridian in \mathcal{E}_2 of $(q_2)^{-1}(T)$, and these are represented in $\Gamma_2 \subset \pi(T)$ by h^2 and $(hxhx^{-2})^2$, respectively.

Since $p': M' \rightarrow M_2$ has degree 6 and $q_{2,1}: M_{4,1} \rightarrow M_2$ has degree 2, the subgroups Γ' and $\Gamma_{2,1}$ corresponding to \mathcal{E}' and $\mathcal{E}_{2,1}$ have indices 6 and 2 in Γ_2 , respectively. Thus the subgroup $\tilde{\Gamma} = \Gamma' \cap \Gamma_{4,1}$ corresponding to $\tilde{\mathcal{E}}$ has index two in Γ' , unless $\Gamma_{4,1}$ contains Γ' . But Lemma 2.2 implies that meridians of $\mathcal{E}(T)$ map to the generator of \mathbb{Z}_4 under $\pi_{n,k}$; hence each has connected preimage in $\mathcal{E}_{4,1}$, represented in $\Gamma_{4,1}$ by h^4 or $(hxhx^{-2})^4$. On the other hand, since the components of

$(q_2)^{-1}(T)$ in M_2 lift to M' , the meridians of $(q_2)^{-1}(T)$ do as well. It follows that Γ' contains the elements h^2 and $(hxx^{-2})^2$ and thus contains $\tilde{\Gamma}$ with index 2. Then $[\Gamma_{4,1} : \tilde{\Gamma}] = 6$, and this is the degree of $p: \tilde{M} \rightarrow M_{4,1}$.

The subgroup $\Lambda < \pi(T)$, representing $\mathcal{E}(T) \cap \partial B^3$, identified in Lemma 2.1 contains the meridian representatives h and hxx^{-2} . Hence the map π_2 determining $q_2: \mathcal{E}_2 \rightarrow \mathcal{E}(T)$ takes Λ onto \mathbb{Z}_2 , so $\mathcal{E}(T) \cap \partial B^3$ has connected preimage in \mathcal{E}_2 . This is the four holed subsurface $\partial \mathcal{E}_2 \cap \partial M_2$, and ∂M_2 is obtained from it by filling with disks of the form $D^2 \times \{\pm 1\}$ bounding components of $(q_2)^{-1}(\mathcal{N}(T))$. Since M' is homeomorphic to $F \times S^1$, it has a single boundary component, and since components of $(q_2)^{-1}(T)$ lift to M' , their meridians lift to \mathcal{E}' . In particular, the subgroup $\Lambda' < \Gamma'$ corresponding to $\partial \mathcal{E}' \cap \partial M'$ contains the meridian representatives h^2 and $(hxx^{-2})^2$ of Γ_2 . Since these elements are not contained in $\Gamma_{4,1}$, $\tilde{\Lambda} = \Lambda' \cap \Gamma_{4,1}$ has index two in Λ' . Therefore $\partial \mathcal{E}' \cap \partial M'$ has connected preimage in $\tilde{\mathcal{E}}$; this is the surface $\tilde{\mathcal{E}} \cap \partial \tilde{M}$. It follows that $\partial \tilde{M}$ is connected.

The boundary of the tetrus, $\partial M_{4,1}$, has genus 3, which can be verified by the fact that it is a fourfold branched cover of ∂B^3 , branched over the four points $T \cap \partial B^3$. Since $\partial \tilde{M} = p^{-1}(\partial M_{4,1})$ is a connected sixfold cover, an Euler characteristic calculation shows that it has genus 13. \square

We now turn from consideration of the manifolds-with-boundary $M_{n,k}$ to their doubles, as defined at the beginning of the paper. It is clear from the definition that a map $f: M \rightarrow N$ between manifolds with boundary determines a map, which we again denote $f: DM \rightarrow DN$, between doubles, and that the map between doubles inherits the property of being a cover or branched cover of degree n from the original map.

Proposition 3.4. *Taking the branched cover $q_{2,1}: DM_{4,1} \rightarrow DM_2$ and the cover $p': DM' \rightarrow DM_2$ to be determined by the corresponding maps on $M_{4,1}$ and M' , respectively, the map supplied by Lemma 1.1 is $\tilde{p}: \tilde{DM} \rightarrow DM_2$.*

The point of this proposition is that Lemma 1.1 is “doubling equivariant”; that is, applying it and then doubling the resulting diagram of (branched) covers yields the same result as doubling first and then applying it. It is a consequence of the normal form theorem for free products with amalgamation.

Fact. Let A and B be groups sharing a subgroup C , and let $A *_C B$ be the free product of A with B , amalgamated over C . If $\pi: A *_C B \rightarrow K$ is an epimorphism to a finite group K such that $\pi(C) = K$, then

$$\ker \pi = \langle \ker \pi|_A, \ker \pi|_B \rangle \simeq (\ker \pi|_A) *_{\ker \pi|_C} (\ker \pi|_B).$$

Proof of Fact. It is clear that $\langle \ker \pi|_A, \ker \pi|_B \rangle$ is contained in $\ker \pi$. We claim equality; this follows from the normal form theorem for free products with amalgamation (see [12, Ch. IV, §2]). Fix sets S_A and S_B of right coset representatives for C in A and B , respectively. Then the normal form theorem asserts that each $g \in A *_C B - \{1\}$ has a *normal form*, which is a unique expression $g = cs_1s_2 \cdots s_n$ for some $n \geq 1$, where $c \in C$ and each s_i is in S_A or S_B , with $s_i \in S_A$ if and only if $s_{i+1} \in S_B$. We will call n the *length* of g , and note that the claim is immediate if g has length 1.

If $g \in \ker \pi$ has length $n > 1$, then write g in normal form as above, and let $c_0 \in C$ have the property that $\pi(c_0) = \pi(cs_1 \cdots s_{n-1})$. Taking $g_0 = cs_1 \cdots s_{n-1}c_0^{-1}$ and

$g_1 = c_0 g_n$, we may write $g = g_0 g_1$ as a product of words in $\ker \pi$. It is evident that g_1 has length 1 and easily proved, by passing elements of C to the left, that g_0 has length at most $n - 1$. Then by induction, each of g_0 and g_1 is in $\langle \ker \pi|_A, \ker \pi|_B \rangle$, so g is as well.

The normal form theorem also implies that the naturally embedded subgroups A and B in $A *_C B$ intersect in C . Therefore $\ker \pi|_A \cap \ker \pi|_B = \ker \pi|_C$, and it follows again from the normal form theorem that the inclusion-induced map $(\ker \pi|_A) *_{\ker \pi|_C} (\ker \pi|_B) \rightarrow \langle \ker \pi|_A, \ker \pi|_B \rangle$ is an isomorphism. \square

Proof of Proposition 3.4. Recall that we have identified the exterior of T , $\mathcal{E}(T)$, with $\overline{B^3 - \mathcal{N}(T)}$, where $\mathcal{N}(T)$ is a regular neighborhood of T in B^3 , with two components homeomorphic to $D^2 \times I$. The double of B^3 is homeomorphic to S^3 , T doubles yielding the link L of Figure 1, and $\mathcal{N}(T)$ doubles yielding a regular neighborhood $\mathcal{N}(L)$. Each component of $\mathcal{N}(L)$ is homeomorphic to $D^2 \times S^1$, with the corresponding component of $\mathcal{N}(T)$, homeomorphic to $D^2 \times I$, mapping in by $(x, y) \mapsto (x, e^{\pi i y})$ and its mirror image by $(x, y) \mapsto (x, e^{-\pi i y})$. Using this description of $\mathcal{N}(L)$, the link exterior $\mathcal{E}(L)$ is the double of $\mathcal{E}(T)$ across the subsurface $\partial B^3 \cap \mathcal{E}(T)$ of $\partial \mathcal{E}(T)$, and meridians of T in $\mathcal{E}(T)$ are meridians of L in $\mathcal{E}(L)$.

Using the above description, van Kampen's theorem describes $\pi(L)$ as a free product with amalgamation, $\pi(L) \simeq \pi(T) *_\Lambda \overline{\pi(T)}$, across the subgroup Λ from Lemma 2.1, which corresponds to $\partial B^3 \cap \mathcal{E}(T)$. Here $\overline{\pi(T)} = \{\bar{g} \mid g \in \pi(T)\}$ is $\pi_1(\overline{\mathcal{E}(T)})$, isomorphic to π_T . There is a “doubling involution” r on $\pi(L)$ determined by $r(g) = \bar{g}$, $g \in \pi(T)$. This is induced by the doubling involution exchanging $\mathcal{E}(T)$ with $\mathcal{E}(T)$ in $\mathcal{E}(L)$, fixing their intersection $\partial B^3 \cap \partial \mathcal{E}(T)$.

The projections $\pi_{n,k}: \pi(T) \rightarrow \mathbb{Z}_n$ (respectively, $\pi_2: \pi(T) \rightarrow \mathbb{Z}_2$) defined in Lemma 2.2 (resp. Lemma 2.3) uniquely determine corresponding projections on $\pi(L)$ with the property that $\pi_{n,k} \circ r = \pi_{n,k}$ (resp. $\pi_2 \circ r = \pi_2$). Since Λ contains the meridian representatives h and $h x h x^{-2}$, each such projection maps it onto the image of $\pi(L)$. Then by the fact above, we have

$$D\Gamma_{n,k} \doteq \ker \pi_{n,k} = \langle \Gamma_{n,k}, \bar{\Gamma}_{n,k} \rangle \simeq \Gamma_{n,k} *_\Lambda \bar{\Gamma}_{n,k},$$

where $\Lambda_{n,k} \doteq \ker \pi_{n,k}|_\Lambda$. (The corresponding fact holds for $D\Gamma_2 \doteq \ker \pi_2$.)

We now define $D\mathcal{E}_{n,k}$ (respectively, $D\mathcal{E}_2$) to be the double of $\mathcal{E}_{n,k}$ (resp. \mathcal{E}_2) across the subsurface $\partial \mathcal{E}_{n,k} \cap \partial M_{n,k} \subset \partial \mathcal{E}_{n,k}$ (resp. $\partial \mathcal{E}_2 \cap \partial M_2$). This notation is somewhat abusive, since this subsurface does not occupy all of $\partial \mathcal{E}_{n,k}$, but we note that it is represented in $\Gamma_{n,k}$ by $\Lambda_{n,k}$, since its complement is $(q_{n,k})^{-1}(\partial \mathcal{N}(T) \cap \mathcal{E}(T))$. Then $D\mathcal{E}_{n,k}$ (respectively, $D\mathcal{E}_2$) covers $\mathcal{E}(L)$, and the description above makes clear that this is the cover corresponding to $D\Gamma_{n,k}$ (resp. $D\Gamma_2$). Hence filling $D\mathcal{E}_{n,k}$ (resp. $D\mathcal{E}_2$) along preimages of meridians of L yields the branched cover $DM_{n,k}$ (resp. DM_2) defined in the statement of the proposition.

We make a similar claim regarding $p': DM' \rightarrow DM_2$. The cover $p': \mathcal{E}' \rightarrow \mathcal{E}_2$ is regular, corresponding to the kernel of a map $\pi': \Gamma_2 \rightarrow \mathbb{Z}_6$, and since $\partial \mathcal{E}' \cap \partial M'$ is connected, it corresponds to a subgroup $\Lambda' = \ker \pi'|_{\Lambda_2}$ of index 6 in Λ_2 . Then defining $\pi': \Gamma_2 *_\Lambda \bar{\Gamma}_2 \rightarrow \mathbb{Z}_6$ by requiring $\pi' \circ r = \pi'$, we argue as above to show that $p': DM' \rightarrow DM_2$ is obtained by filling the cover of $D\mathcal{E}_2$ corresponding to $\ker \pi'$ along preimages of meridians.

We recall from the proof of Lemma 3.3 that $\tilde{\Gamma} = \Gamma_{4,1} \cap \Gamma'$ has index 2 in Γ' , and that $\tilde{\Lambda} = \Gamma_{4,1} \cap \Lambda'$ has index 2 in Λ' . Then it follows as above that the subgroup

$D\tilde{\Gamma} \doteq D\Gamma_{4,1} \cap D\Gamma' = \langle \tilde{\Gamma}, r(\tilde{\Gamma}) \rangle$, and that it is isomorphic to the free product of $\tilde{\Gamma}$ with itself, amalgamated across $\tilde{\Lambda}$. The proposition follows. \square

Proposition 3.4 supplies a diamond of maps to which Proposition 1.2 may be applied. We thus prove Theorem 0.1 below by using Proposition 1.3 to find a fibering of M' transverse to the preimage of L .

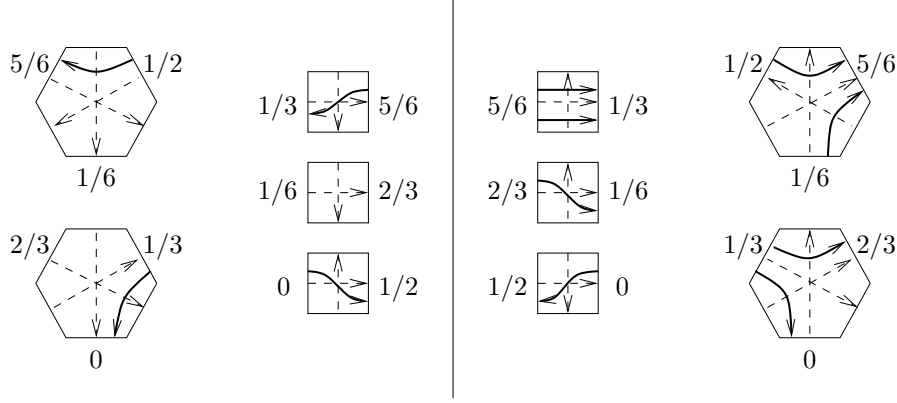
Proof of Theorem 0.1. By Lemma 3.2, M_2 is homeomorphic to a bundle over S^1 with fiber the surface F depicted in Figure 5 and monodromy $\sigma: F \rightarrow F$. The fibers of M_2 join to yield a fibering of DM' with fiber surface DF , the double of F , and monodromy map which we will call $D\sigma$. This is pictured in Figure 6. Here the hexagons and squares to the left of the vertical line are fitted together along labeled arcs as in Figure 5 forming a copy of F , and the hexagons and squares to the right of the vertical line are fitted together along their labeled edges forming a copy of \bar{F} . To form DF , each unlabeled edge is identified with its correspondent by reflection through the vertical line. $D\sigma$ is the map that restricts on F to σ and is equivariant with respect to the doubling involution, hence is itself of order 6.

Recall that $p': M' \rightarrow M_2$, defined below Lemma 3.2, is the sixfold cover of M_2 which is trivially fibered over S^1 with fiber F . Since DM' is the double of M' , it is trivially fibered over S^1 with fiber DF . Using Lemma 3.2, we may identify DM_2 with $DF \times I / ((x, 0) \sim (D\sigma(x), 1))$. Then identifying DM' with $DF \times I / ((x, 0) \sim (x, 1))$, the covering map is given by $(x, y) \mapsto ((D\sigma)^k(x), 6(y - \frac{k}{6}))$ for $y \in [\frac{k}{6}, \frac{k+1}{6}]$, $0 \leq k < 6$. Since the components of $(q_2)^{-1}(T)$ lie in disjoint copies of the fiber surface F for M_2 , the components of $q_2^{-1}(L)$ lie in disjoint copies of $DF \subset M$. Take $\pi: DM' \rightarrow DF$ to be projection onto the first factor. Then by the description above, if λ is a component of $(q_2 \circ p')^{-1}(L) \subset M'$, $\pi(\lambda)$ is a simple closed curve on DF and $D\sigma$ takes $\pi(\lambda)$ to $\pi(\lambda')$, where λ' is another component of $(q_2 \circ p')^{-1}(L)$.

To understand $(q_2 \circ p')^{-1}(L)$, we first describe $(q_2)^{-1}(T)$ in M_2 . This is the fixed set of the involution which restricts on each of $V_{1/3}$ and $V_{1/2}$ to the complex conjugation-induced involution. The fixed arc $[1, 1] \times \{1\} \subset V_{1/3}$ lies in the disk H_0 , running from the midpoint of the side labeled 0 to the midpoint of the opposite, unlabeled side in Figure 5. The other arc $[-1, 1] \times \{-1\} \subset V_{1/3}$ runs from the midpoint of the side of H_1 labeled $1/2$ to the midpoint of the opposite side. The arc $[-1, 1] \times \{1\} \subset V_{1/2}$ lies in S_0 , joining the midpoints of the sides labeled 0 and $1/2$. Thus the union of these three arcs is an arc properly embedded in M_2 , comprising one component of the preimage of T .

The other component of $q_2^{-1}(T)$ is the arc $[-1, 1] \times \{-1\} \subset V_{1/2}$, with endpoints in $B_{1/2} \subset \partial M_2$. This lies in the disk $D^2 \times \{-1\}$ in $V_{1/2}$, midway between S_1 and $\sigma(S_1) = S_2$. Thus using the description from Lemma 3.2 of M_2 as $F \times I / (x, 0) \sim (\sigma(x), 1)$, the second arc of $q_2^{-1}(T)$ lies in $S_1 \times \{1/2\}$, joining midpoints of the unlabeled boundary components.

Now in DM_2 , $(q_2)^{-1}(L)$ is the double of $(q_2)^{-1}(T)$, with one component in $DF \times \{0\}$ and one in $DF \times \{1/2\}$. Then $(q_2 \circ p')^{-1}(L) \subset DM'$ has twelve components in disjoint copies of DF . This set is depicted by the dashed arcs in Figure 6. If α is such an arc, a simple closed curve on DF containing α may be obtained by taking the union of a collection of arcs \mathcal{A} , of minimal cardinality such that $\alpha \in \mathcal{A}$ and for each $\beta \in \mathcal{A}$, the arcs meeting β at its endpoints are also in \mathcal{A} . Inspection reveals six such simple closed curves, permuted by $D\sigma$, each of the form $\pi(\lambda)$ for

FIGURE 6. The collection of transverse curves in DF

some component λ of $(q_2 \circ p')^{-1}(L)$. There are six, rather than twelve, because $(D\sigma)^3$ is the hyperelliptic involution of DF , which preserves simple closed curves.

The bold arcs depicted in Figure 6 join to produce two disjoint simple closed curves in DF transverse to this collection. We have indicated orientations with arrows so that, giving DF the standard orientation from the page, the algebraic and geometric intersection numbers of each bold curve with each dashed curve coincide. Thus by Proposition 1.3, spinning annuli along the bold curves yields a fibration of DM' transverse to $(q_2 \circ p')^{-1}(L)$. Then by Proposition 1.2, \widetilde{DM} is fibered, and the branched cover $q: \widetilde{DM} \rightarrow DM'$ takes fibers to fibers.

If \widetilde{F} is the fiber surface of \widetilde{DM} , then q restricts on \widetilde{F} to a twofold branched cover of a fiber surface of DM' transverse to $(q_2 \circ p')^{-1}(L)$, branched over their points of intersection. From Figure 6, we find 16 points of intersection between the arcs of $\pi((q_2 \circ p')^{-1}(L))$ and the bold arcs which determine the spinning curves. It follows that a spun fiber surface has 32 points of intersection with $(q_2 \circ p')^{-1}(L)$, since π maps these curves two-to-one. Since the spun fiber surface of DM' has genus two, an Euler characteristic calculation shows that \widetilde{F} has genus 19. Together with the information in Lemma 3.3 and Proposition 3.4, this establishes the theorem. \square

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